Permutation groups and transformation semigroups

Peter J. Cameron, University of St Andrews

Abstract

I have been working on this topic for fifteen years now, since being introduced to it by João Araújo. His thesis is that, rather than the situation where semigroup theorists simply reduce their problem to group theory and hand it over to the group theorists, it is much better to have a dialogue between the two areas. Typically, an argument combining group and semigroup theory (and maybe combinatorics) is needed to establish strong results about semigroups, but in addition, semigroups suggest exciting new areas of permutation group theory to work on.

The key concept in permutation group theory is primitivity, going back to the work of Galois: a permutation group is primitive if it preserves no non-trivial partition of its domain. The O'Nan–Scott Theorem together with the classification of finite simple groups gives us a lot of information about primitive groups. But open problems remain.

I will begin with a run through primitivity, multiple transitivity, and some of the related concepts to have arisen recently. Then I will turn to concepts specifically related to transformation semigroups. These fall into two classes:

Synchronization: This topic begins in automata theory, motivated by the Černý conjecture. The most general form for which we have strong results is this: for which permutation groups G of degree n is it true that, for any non-permutation f, the semigroup generated by G and f contains a map of rank 1? This is equivalent to saying that G preserves no non-trivial *weakly perfect* graph.

Regularity and idempotent generation: For which permutation groups G is it the case that, if f is any map of rank k, then $\langle G, f\rangle$ is regular, or is idempotent-generated? If $k > 2$ this property resembles multiple transitivity. But for $k = 2$, the property for regularity is exactrly equivalent to primitivity, whereas idempotent generation is equivalent to a strengthening of primitivity called the *road closure property.* Work is in progress to understand this.